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Fp-Projective and Fp-Cotorsion Modules

Shhaied Riad *

Department of University Requirements-Faculty of Science and Arts, International University for Science and Technology, Damascus, Syria
E-mail: rshhaied@yahoo.com

Abstract

Let R be a ring. The concepts of fp-projective and fp-cotorsion R-modules are defined. These modules together with the concept of the cotorsion theory are used to characterize left Noetherian rings, left perfect and right coherent rings, left coherent rings and left fp-regular rings. Moreover, some known characterizations of von Neumann regular rings are found.

Keywords: Cotorsion theory; fp-projective modules; fp-cotorsion modules; Noetherian rings; perfect rings; coherent rings; fp-regular rings.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules considered are unitary. For any R-module M, E(M) is the injective envelope of M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. R-Mod (Mod-R) denotes the category of all left (right) R-modules. Hom(M, N) (resp., $\mathrm{Ext}^1(M, N)$, $\mathrm{Tor}_1(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp., $\mathrm{Ext}_R^1(M, N)$, $\mathrm{Tor}_1^R(M, N)$).

Recall that a left R-module M is FP-injective [1], if $\mathrm{Ext}^1(A, M) = 0$ for every finitely presented left R-module A, M is F-injective [2] if $\mathrm{Ext}^1(R/I, M) = 0$ for every finitely generated left ideal I, M is fp-injective if $\mathrm{Ext}^1(R/I, M) = 0$ for every finitely presented left ideal I. A right R-module M is fp-flat, if $\mathrm{Tor}_1(M, R/I) = 0$ for every finitely presented left ideal I. fp-injective (or almost F-injective) and fp-flat (or almost F-flat) modules have been introduced and studded in [3].

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* Corresponding author.

Let M be a left R-module. Then M is FP-projective (F-projective), if $\mathrm{Ext}^1(M, N) = 0$ for any FP-injective (F-injective) left R-module N, M is cotorsion if $\mathrm{Ext}^1(N, M) = 0$ for any flat left R-module N. More information for the concepts of FP-projective, F-projective and cotorsion modules can be found in [4], [5] and [6].

Given a class $\mathcal{C} \subseteq$ Mod-R, then $\mathcal{C}^{\perp} = \{N \in \text{Mod-R} \mid \mathrm{Ext}^1(C, N) = 0 \text{ for all } C \in \mathcal{C}\}$, ${}^{\perp}\mathcal{C} = \{N \in \text{Mod-R} \mid$

$\mathrm{Ext}^1(N, C) = 0 \text{ for all } C \in \mathcal{C}\}$. When $\mathcal{C} \subseteq$ R-Mod, $\mathcal{C}^{\perp}$ and ${}^{\perp}\mathcal{C}$ can be defined similarly.

Let $\mathcal{C}$ be a class of left (right) R-modules and M be a left (right) R-module. A homomorphism $\varphi: M \to F$ where $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope [7] of M if for any homomorphism $f: M \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g: F \to F'$ such that $g\varphi = f$. If the only such g are automorphisms of F when $F' = F$ and $f = \varphi$, then the $\mathcal{C}$-preenvelope φ is called a $\mathcal{C}$-envelope of M.

Following [7, Definition 7.1.6], a monomorphism $\alpha: M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of M if coker $(\alpha) \in {}^{\perp}\mathcal{C}$. Dually we have the definitions of $\mathcal{C}$-precover, special $\mathcal{C}$-precover and -cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let $\mathcal{A}$, $\mathcal{B} \subseteq$ Mod-R. The pair $(\mathcal{A}, \mathcal{B})$ is called a cotorsion theory [7], if $\mathcal{A}^{\perp} = \mathcal{B}$ and ${}^{\perp}\mathcal{B} = \mathcal{A}$. The cotorsion theory $(\mathcal{A}, \mathcal{B})$, when $\mathcal{A}, \mathcal{B} \subseteq$ R-Mod can be defined similarly.

A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is generated (cogenerated) by a class $\mathcal{C} \subseteq$ Mod-R, if $\mathcal{A} = {}^{\perp}\mathcal{C}$ ($\mathcal{B} = \mathcal{C}^{\perp}$). A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is said to be perfect (complete) if every right R-module has a $\mathcal{B}$-envelope and an $\mathcal{A}$-cover (a special $\mathcal{B}$-preenvelope and a special $\mathcal{A}$-precover) (see [8] and [9]). A cotorsion theory $(\mathcal{A}, \mathcal{B})$ is said to be hereditary [8] if whenever $0 \to K \to P \to F \to 0$ is exact with $P, F \in \mathcal{A}$, then K is also in $\mathcal{A}$. By [8, Proposition 1.2], $(\mathcal{A}, \mathcal{B})$ is hereditary if and only if $0 \to A \to B \to C \to 0$ is exact with $A, B \in \mathcal{B}$, then C is also in $\mathcal{B}$.

In section two of this note, we first extend the concepts of F-projective modules and cotorsion modules to fp-projective modules and fp-cotorsion modules respectively, and then give some characterizations and properties of fp-injective modules and fp-projective modules. We also prove that (fp-proj, fp-inj) is a complete cotorsion theory, and (fp-fl, fp-cot) is a perfect cotorsion theory, where fp-proj (fp-inj) denotes the class of all fp-projective (fp-injective) modules, and fp-fl (fp-cot) the class of all fp-flat (fp-cotorsion) modules.

The activity of section two will be appeared in section three, where we present some characterizations of left Noetherian rings, left coherent and right perfect rings, left coherent rings, left fp-regular [3] rings by fp-projective and fp-cotorsion R-modules. Moreover, some known characterizations of von Neumann regular rings are found.

2. Definitions and first results

We first introduce the concepts of fp-projective and fp-cotorsion modules.

2.1 Definition

Let M be a left R-module. Then we call

(1) M is fp-projective if $\mathrm{Ext}^1(M, N) = 0$ for any fp-injective left R-module N.

(2) M is fp-cotorsion if $\mathrm{Ext}^1(N, M) = 0$ for any fp-flat left R-module N.

The right version can be defined similarly.

For any ring R, it is clear that every projective left R-module is fp-projective, R/I is fp-projective left R-module for every finitely presented left ideal I, every fp-projective left R-module is F-projective and FP-projective, every injective left R-module is fp-cotorsion and every fp-cotorsion left R-module is cotorsion.

Next, we give some characterizations of fp-injective R-modules by fp-projective R-modules.

2.2 Theorem

Let M be a left R-module. The following statements are equivalent.

(1) M is fp-injective left R-module.
(2) M is injective with respect to every exact sequence $0 \to C \to B \to A \to 0$ of left R-modules with A fp-projective.
(3) M is injective with respect to every exact sequence $0 \to K \to P \to A \to 0$ of left R-modules with P projective and A fp-projective.
(4) Every fp-projective R-module is projective with respect to every exact sequence of left R-modules $0 \to M \to M' \to M'' \to 0$.
(5) Every fp-projective left R-module is projective with respect to the canonical exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$.
(6) There exists an exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' injective, such that every fp-projective left R-module is projective with respect to this sequence.
(7) There exists an exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' fp-injective, such that every fp-projective left R-module is projective with respect to this sequence.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1). Let I be a finitely presented left ideal of R. We have the exact sequence $0 \to I \to R \to R/I \to 0$, which induces the exact sequence $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M) \to \mathrm{Ext}^1(R/I, M) \to 0$. Since R/I is fp-projective, $\mathrm{Hom}(R, M) \to \mathrm{Hom}(I, M) \to 0$ is exact by (3). So $\mathrm{Ext}^1(R/I, M) = 0$ and so (1) follows.

(1) $\Rightarrow$ (4). Let M be an fp-injective left R-module. Then the sequence $\mathrm{Hom}(P, M') \to \mathrm{Hom}(P, M'') \to \mathrm{Ext}^1(P, M) = 0$ is exact for every fp-projective left R-module P, and so (4) follows.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(7) $\Rightarrow$ (1). Assume that $0 \to M \to M' \to M'' \to 0$ is exact sequence of left R-modules with M' fp-injective and, I is any finitely presented left ideal of R. Since R/I is fp-projective left R-module, then by (7), the sequences $\mathrm{Hom}(R/I, M') \to \mathrm{Hom}(R/I, M'') \to \mathrm{Ext}^1(R/I, M) \to \mathrm{Ext}^1(R/I, M') = 0$ and $\mathrm{Hom}(R/I, M') \to \mathrm{Hom}(R/I, M'') \to 0$ are exact. This implies that $\mathrm{Ext}^1(R/I, M) = 0$, and (1) follows. □

Now, we give some characterizations of fp-projective R-modules by fp-injective R-modules.

2.3 Theorem

Let M be a left R-module. The following statements are equivalent.

1. M is left fp-projective.
2. M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with A fp-injective.
3. M is projective with respect to every exact sequence $0 \to A \to F \to K \to 0$ of R-modules with F injective and A fp-injective.
4. Every fp-injective R-module is injective with respect to every exact sequence of left R-modules $0 \to M'' \to M' \to M \to 0$.
5. There exists an exact sequence $0 \to M'' \to M' \to M \to 0$ of left R-modules with M' projective, such that every fp-injective left R-module is injective with respect to this sequence.
6. There exists an exact sequence $0 \to M'' \to M' \to M \to 0$ of left R-modules with M' fp-projective, such that every fp-injective left R-module is injective with respect to this sequence.

Proof. (1) $\Rightarrow$ (2). Let M be an fp-projective left R-module. Then the sequence $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \mathrm{Ext}^1(M, A) = 0$ is exact for every fp-injective R-module A. Hence (2) holds.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Let $0 \to A \to F \to K \to 0$ be an exact sequence of left R-modules with F injective and A fp-injective. Then the sequence $\mathrm{Hom}(M, F) \to \mathrm{Hom}(M, K) \to 0$ is exact by (3). So $\mathrm{Ext}^1(M, A) = 0$ for any fp-injective left R-module A. Hence M is fp-projective left R-module.

(1) $\Rightarrow$ (4). Let M be an fp-projective left R-module, F be an fp-injective left R-module. Then the sequence $\mathrm{Hom}(M', F) \to \mathrm{Hom}(M'', F) \to \mathrm{Ext}^1(M, F) = 0$ is exact. So (4) holds.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1). Let $0 \to M'' \to M' \to M \to 0$ be an exact sequence of left R-modules with M' fp-projective. Then for any fp-injective left R-module N, the sequence $\mathrm{Hom}(M', N) \to \mathrm{Hom}(M'', N) \to \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M', N) = 0$ is exact, and the sequence $(M', N) \to \mathrm{Hom}(M'', N) \to 0$ is exact by (6).This implies that $\mathrm{Ext}^1(M, N) = 0$, and (1)

follows. □

A cotorsion theory ($\mathcal{A}$, $\mathcal{B}$) is said to have enough injectives (resp., enough projectives) (see. [10, p. 5]) if for every module L there is a short exact sequence $0 \to L \to B \to A \to 0$ (resp., $0 \to B \to A \to L \to 0$) such that B ∈ $\mathcal{B}$ and A ∈ $\mathcal{A}$.

2.4 Lemma

[10, Theorem 10]. Every cotorsion theory which is cogenerated by a set of modules has enough projectives and enough injectives. □

2.5 Lemma

(see [7, p.159]). If a cotorsion theory ($\mathcal{A}$, $\mathcal{B}$) has enough projectives and enough injectives, then every module has a special $\mathcal{B}$-preenvelope and a special $\mathcal{A}$-precover. □

In what follows, we assume that fp-proj (fp-inj) denotes the class of all fp-projective (fp-injective) left R-modules, and fp-fl (fp-cot) denotes the class of all fp-flat (fp-cotorsion) right R-modules.

2.6 Theorem

Let R be a ring. Then (fp-proj, fp-inj) is a complete cotorsion theory.

Proof. Let $\mathcal{C}$ ={R/I | I is finitely presented left ideal I of R}. Then $\mathcal{C}^{\perp}$={N ∈ R-Mod | Ext^1(R/I, N)} = 0 for any finitely presented left ideal I of R}, so $\mathcal{C}^{\perp}$ = fp-inj. On the other hand, we have $^{\perp}(\mathcal{C}^{\perp})$ = {N ∈ R-Mod | Ext^1(N, C) = 0 for all C ∈ $\mathcal{C}^{\perp}$}, so $^{\perp}(\mathcal{C}^{\perp})$ = fp-proj. Now we have $\mathcal{C}^{\perp} = (^{\perp}(\mathcal{C}^{\perp}))^{\perp}$. It follows that the pair (fp-proj, fp-inj) is a cotorsion theory that is cogenerated by the class $\mathcal{C}$. So the result follows from Lemma 2.4 and Lemma 2.5 □

Following [9], for a class of (right, resp. left) R-modules $\mathcal{C}$,we put

$\mathcal{C}^{\top}$ = {N ∈ R-Mod | Tor_1(C, N) = 0 for all C ∈ $\mathcal{C}$}, resp.

$^{\top}\mathcal{C}$ = {N ∈ Mod-R | Tor_1(N, C) = 0 for all C ∈ $\mathcal{C}$}.

Let $\mathcal{A}$ ⊆ Mod-R, $\mathcal{B}$ ⊆ R-Mod. The pair ($\mathcal{A}$, $\mathcal{B}$) is called a Tor-torsion theory if $\mathcal{A} = {}^{\top}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\top}$.

In what follows, by $\mathcal{PI}$ we denote the class of all pure injective right R-modules.

2.7 Lemma

[see. 9, Lemma 1.11]. Let $\mathcal{A}$ ⊆ Mod-R, $\mathcal{B}$ ⊆ R-Mod and ($\mathcal{A}$, $\mathcal{B}$) be a Tor-torsion theory. Then ($\mathcal{A}$, $\mathcal{A}^{\perp}$) is a cotorsion theory generated by $\mathcal{C}$ = {M^+ | M ∈ $\mathcal{B}$} ⊆ $\mathcal{PI}$. □

2.8 Lemma

[see.9, Theorem 2.8]. Let R be a ring and ($\mathcal{A}$, $\mathcal{B}$) be a cotorsion theory generated by a class $\mathcal{C} \subseteq \mathcal{PI}$. Then ($\mathcal{A}$, $\mathcal{B}$) is complete. Moreover, $\mathcal{A}$ is a cover class and $\mathcal{B}$ is an envelope class. □

2.9 Theorem

Let R be a ring. Then (fp-fl, fp-cot) is a perfect cotorsion theory.

Proof. Let $\mathcal{B}$ ={R/I | I is finitely presented left ideal I of R}, ${}^{\top}\mathcal{B}$ = {N ∈ Mod-R | Tor_1(N, R/I) = 0 for any finitely presented left ideal I of R}. Let $\mathcal{A}$ = fp-fl. Then we have $\mathcal{A}^{\top}$ = {N ∈ R-Mod | Tor_1(M, N) = 0 for any fp-flat right R-module M}. Since M is a fp-flat right R-module if and only if Tor_1(M, R/I) = 0 for every finitely presented left ideal I, $\mathcal{A} = {}^{\top}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\top}$. So ($\mathcal{A}$, $\mathcal{B}$) is a Tor-torsion theory. By lemma 2.7, ($\mathcal{A}, \mathcal{A}^{\perp}$) is a cotorsion theory generated by $\mathcal{C}$ = {M^+ | M ∈ $\mathcal{B}$} ⊆ $\mathcal{PI}$. By lemma 2.8, ($\mathcal{A}, \mathcal{A}^{\perp}$) is a perfect cotorsion theory. Finally we note that $\mathcal{A}^{\perp}$ = {N ∈ Mod-R | Ext^1(M, N) = 0, for any fp-flat right R-module M} = fp-cot. Thus (fp-fl, fp-cot) is a perfect cotorsion theory, as desired. □

3. Characterizations of some classes of rings

It is well known that a ring R is left Noetherian if and only if every F-injective left R-module is injective [11, Corollarly1.10]. Next, we characterize left Noetherian rings by fp-injective left R-modules and fp-projective left R-modules.

3.1 Theorem

For any ring R, the following statements are equivalent.

(1) R is left Noetherian.
(2) Every left R-module is fp-projective.
(3) Every finitely generated left R-module is fp-projective.
(4) Every cyclic left R-module is fp-projective.
(5) Every fp-injective left R-module is injective.
(6) (fp-proj, fp-inj) is hereditary, and every fp-injective left R-module is fp-projective.

Proof. (2) ⇒ (3) ⇒ (4) are clear.

(4) ⇒ (5). Let M any fp-injective left R-module and I any left ideal of R. Then Ext^1(R/I, M) = 0 by (4).Thus M is injective, as required.

(5) ⇒ (2). Let M any left R-module, 0 → A → B → C → 0 be an exact sequence with A fp-injective. Since A is injective by (5), the sequence Hom(M, B) → Hom(M, C) → Ext^1(M, A) = 0 is exact. So M is fp-projective by Theorem 2.3 (2).

(5) ⇒ (1). It is clear that every F-injective left R-module is fp-injective. So by (5) and [11, Corollary1.10], R is left Noetherian.

(1) ⇒ (5). Let R be a left Noetherian ring. Then every left ideal of R is finitely presented. So every fp-injective left R-module is F-injective. Thus (5) holds by [11, Corollary1.10].

(2) ⇒ (6) is clear.

(6) ⇒ (2). Let M any left R-module. Since (fp-proj, fp-inj) is a complete cotorsion theory by Theorem 2.6 , M has a special fp-injective preenvelope, i.e. there is a short exact sequence $0 \to M \to F \to P$, where F is fp-injective and P is fp-projective. Since P is fp-projective, and (fp-proj, fp-inj) is hereditary and F is fp-projective by (6), M must be fp-projective. Thus (2) holds. □

Recall that a ring R is called left coherent if every finitely generated left ideal of R is finitely presented. Coherent rings and their generations have been studied and characterized by many authors (see. for example [1 ,3,5,6,12,13]).

Next, we give some characterizations of left coherent rings in terms of fp-projective and fp-cotorsion left R-modules.

3.2 Theorem

The following statements are equivalent for a ring R.

(1) R is left coherent.
(2) Every fp-injective left R-module is FP-injective.
(3) Every fp-injective left R-module is F-injective.
(4) Every fp-flat right R-module is flat.
(5) Every FP-projective left R-module is fp-projective.
(6) Every finitely presented left R-module is fp-projective.
(7) Every F-projective left R-module is fp-projective.
(8) For every finitely generated left ideal I of R, R/I is fp-projective.
(9) Every cotorsion right R-module is fp-cotorsion.

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) by [3, theorem 2.8].

(5) ⇒ (6) is clear, since every finitely presented left R-module is FP-projective.

(6) ⇒ (2). Let M be an fp-injective left R-module and A any finitely presented left R-module. Then $\mathrm{Ext}^1(A, M) = 0$ by (6).Thus M is FP-injective.

(2) ⇒ (5). Let M be an FP-projective left R-module and N any fp-injective left R-module. Since N is FP-

injective by (2), $\mathrm{Ext}^1(M, N) = 0$. Thus M is fp-projective.

(3) ⇒ (7). Let M be an F-projective left R-module and N any fp-injective left R-module. Since N is F-injective by (3), $\mathrm{Ext}^1(M, N) = 0$. Thus M is fp-projective.

(7)⇒(8) since R/I is F-projective left R-module for any finitely generated left ideal I of R.

(8) ⇒ (3). Let M be an fp-injective left R-module and I any finitely generated left ideal. Then $\mathrm{Ext}^1(R/I, M) = 0$ by (8). Thus M is F-injective.

(4) ⇒ (9). Let M be a cotorsion right R-module and N any fp-flat right R-module. Since N is flat by (4), $\mathrm{Ext}^1(N, M) = 0$. So M is fp-cotorsion.

(9) ⇒ (4) is similar to (4) ⇒ (9). □

Recall that a ring R is called right perfect if every left R-module has a projective cover [14]. A classical theorem of Bass states that R is right perfect if and only if every flat right R-module is projective [15, Theorem and Definition 2.1]. Using this fact with the Theorem 3.2, we get the following characterizations of left coherent and right perfect ring in terms of fp-injective and fp-cotorsion modules.

3.3 Theorem

The following statements are equivalent for a ring R.

1. R is left coherent and right perfect.
2. Every right R-module M is fp-cotorsion.
3. (fp-fl, fp-cot) is hereditary and every fp-flat right R-module is fp-cotorsion .

Proof. (1) ⇒ (2). Let M be a right R-module and N any fp-flat right R-module. Since R is left coherent, N is flat right R-module by [3, Theorem 2.8 (4)]. Since R is right perfect, N is projective right R-module, so $\mathrm{Ext}^1(N, M) = 0$. Hence M is fp-cotorsion.

(2) ⇒ (1). Since every cotorsion right R-module is fp-cotorsion by (2), R is left coherent by Theorem 3.2 (9). On the other hand, if N is a flat right R-module, then for any right R-module M, $\mathrm{Ext}^1(N, M) = 0$ by (2). So N is projective right R-module and hence R is right perfect.

(2) ⇒ (3) is clear.

(3) ⇒ (2). Let M be any right R-module. Since (fp-fl, fp-cot) is a perfect cotorsion theory, M has an fp-flat cover by Theorem 2.9, i.e. there exists an exact sequence $0 \to L \to F \to M \to 0$ with F fp-flat right R-module and L fp-cotorsion right R-module. By (3) F is fp-cotorsion right R-module, and so M must be fp-cotorsion. Thus (2) holds. □

It is well known that a ring R is a von Neumann regular ring if and only if every finitely generated left ideal of R is a direct summand of R. Regular rings and their generations have been studied and characterized by many authors (see. for example [3,5,6,8,16]).

Following [3, p. 124], we call a ring R, a left fp-regular (or left almost regular), if every finitely presented left ideal of R is a direct summand of R. The right fp-regular ring can be defined similarly. Next, we give some characterizations of left fp-regular rings in terms of fp-projective and fp-cotorsion R-modules.

3.4 Theorem

For any ring R, the following statements are equivalent.

(1) R is a left fp-regular.
(2) For any finitely presented left ideal I, R/I is projective left R-module.
(3) Every left R-module is fp-injective.
(4) Every right R-module is fp-flat.
(5) Every fp-projective left R-module is projective.
(6) Every cotorsion left R-module is fp-injective.
(7) Every fp-cotorsion right R-module is injective.
(8) Every cotorsion right R-module is fp-flat.
(9) (fp-fl, fp-cot) is hereditary and every fp-cotorsion right R-module is fp-flat.
(10) (fp-proj, fp-inj) is hereditary and every fp-projective left R-module is fp- injective.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) by [3, Theorem 3.5].

(3) $\Rightarrow$ (5). Let M any fp-projective left R-module. Then by (3), $\mathrm{Ext}^1(M, N) = 0$ for any left R-module N, so M is projective.

(5) $\Rightarrow$ (2) is clear, since R/I is fp-projective left R-module for every finitely presented left ideal I.

(3) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (2). Let I any finitely presented left ideal of R, M any cotorsion left R-module. We have M is fp-injective by (6), and R/I is fp-projective left R-module. So M is injective with respect to the exact sequence $0 \to I \to R \to R/I \to 0$ of left R-modules by Theorem 2.2 (2), and so $\mathrm{Ext}^1(R/I, M) = 0$. By the arbitrariness of the cotorsion left R-module M, it follows that R/I is flat. Thus R/I is projective since R/I is finitely presented.

(4) $\Rightarrow$ (7). Let M be an fp-cotorsion right R-module. Then, by (4) $\mathrm{Ext}^1(N, M) = 0$ for any right R-module N, so M is injective right R-module.

(7) $\Rightarrow$ (4). Let N be a right R-module, M any fp-cotorsion right R-module. Then, by (7) $\mathrm{Ext}^1(N, M) = 0$, so N is fp-flat right R-module.

(4) $\Rightarrow$ (8) is clear.

(8) $\Rightarrow$ (3). Let M be any left R-module. Then M^+ is pure-injective right R-module and hence M^+ is a cotorsion right R-module. So M^+ is fp-flat right R-module by (8), and so M is fp-injective left R-module by [3, Theorem 2.7 (1)].

(4) $\Rightarrow$ (9) and (3) $\Rightarrow$ (10) are obvious.

(9) $\Rightarrow$ (4) Let M be any right R-module. By Theorem 2.9 there exists an exact sequence $0 \to M \to C \to L \to 0$ with C fp-cotorsion right R-module and L fp-flat right R-module. So C is fp-flat right R-module by (9). Since (fp-fl, fp-cot) is hereditary M is fp-flat right R-module.

(10) $\Rightarrow$ (3). Let M be any left R-module. By Theorem 2.6 there exists an exact sequence $0 \to N \to P \to M \to 0$, where P is fp- projective left R-module and N is fp-injective left R-module. So P is fp-injective by (10).Since (fp-proj, fp-inj) is hereditary, M is fp-injective. □

4. Conclusion

It is not known whether, the left fp-regular right coherent ring is von Neumann regular ring. On the other hand , it is clear that every von Neumann regular ring is right and left fp-regular ring, but the converse is not true as shown by [3, Example 3.6].

By Theorem 3.2(4) and Theorem 3.4(4), it is obvious that R is a von Neumann regular ring if and only if R is a left fp-regular left coherent**.** Using this fact with Theorem 3.2 and Theorem 3.4, we can get the following known characterizations of von Neumann regular ring:

(1) R is von Neumann regular ring.
(2) Every right R-module is flat.
(3) every left R-module is F-injective.
(4) every cotorsion left R-module is F-injective.
(5) every cotorsion right R-module is flat.
(6) Every F-projective left R-module is projective.
(7) (F-proj, F-inj) is hereditary and every F-projective left R-module is F-injective. □

Note that the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) have been proved by other ways in [11, Theorem 1.11], (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) in [17, Theorem 3.2.2]. and (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) in [6, Theorem 3.2].

5. Recommendations

It is well known that a ring R is quasi-Frobenius when every left R-module embeds in a free left R-module. If the embedding is restricted to finitely generated left R-modules, then R is called left FGF ring, and the question of whether any left FGF ring is QF is known as the FGF conjecture (or Faith's Problem). This conjecture is still an open question. But there are many affirmative answers to this question with different additional conditions. (see [18]–[20]). It is recommended to characterize the quasi-Frobenius rings in terms of fp-projective and fp-cotorsion left R-modules, and then reach a new affirmative answers to the FGF conjecture with additional condition.

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